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Solutions for a Cosmic String

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For a conformally flat metric $ds^2 = a^2(\eta)(d\eta^2 - dx^2 - dy^2 - dz^2)$ Vilenkin obtained the equation

$$\frac{\partial}{\partial\eta}\left[\frac{a^2(\eta)\,\dot{y}}{\sqrt{1 + y'^2 - \dot{y}^2}}\right] = \frac{\partial}{\partial x}\left[\frac{a^2(\eta)y'}{\sqrt{1 + y'^2 - \dot{y}^2}}\right]$$

for a cosmic string and gave some particular solutions both for a = const and $a \neq$ const. The present work completely solves the equation for a = const and extend the work of Vilenkin for $a \neq$ const.

Vilenkin[(1)] obtained an equation for a cosmic string in a conformally flat space-time given by

$$ds^2 = a^2(\eta)(d\eta^2 - dx^2 - dy^2 - dz^2), \qquad a(\eta)\, d\eta = dt$$

Vilenkin gave some particular solutions for the flat-space-time case where $a(\eta) = 1$ and also indicated some approximate solutions for a radiation-dominated universe where $a(\eta) \simeq \eta$. In this paper we solve completely the equations for the flat-space-time case where $a(\eta) = 1$ and obtain a particular solution for the expanding-universe case with arbitrary $a(\eta)$.

For a conformally flat-space-time cosmological model given by

$$ds^2 = a^2(\eta)(d\eta^2 - dx^2 - dy^2 - dz^2) \tag{1}$$

where $a(\eta)\, d\eta = dt$, Vilenkin obtained the following equation for a cosmic string:

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$$\frac{\partial}{\partial \eta}\left[\frac{a^2(\eta)\dot{y}}{\sqrt{1 + y'^2 - \dot{y}^2}}\right] = \frac{\partial}{\partial x}\left[\frac{a^2(\eta)y'}{\sqrt{1 + y'^2 - \dot{y}^2}}\right] \tag{2}$$

where $\dot{y} = \partial y(\eta, x)/\partial \eta$ and $y' = \partial y(\eta, x)/\partial x$.

Vilenkin noted that a class of particular solutions of equation (2) for $a(\eta) = 1$ is given by

$$y = f(x \pm \eta)$$

where f is an arbitrary function.

Vilenkin also noted that for a radiation-dominated universe one gets $a(\eta) \simeq \eta$. He then indicated some approximate solutions for $a(\eta) = \eta$ and $\dot{y}, y' << 1$. The present work extends the work of Vilenkin. To this end we note that a necessary and sufficient condition for $y(x, \eta)$ to satisfy equation (2) is that there exist $z(x, \eta)$ such that

$$\frac{a^2(\eta)\dot{y}}{\sqrt{1 + y'^2 - \dot{y}^2}} = z' \tag{3a}$$

$$\frac{a^2(\eta)y'}{\sqrt{1 + y'^2 - \dot{y}^2}} = \dot{z} \tag{3b}$$

specifically, the present work seeks to obtain solutions of equation (2) by obtaining solutions of equations (3). For this it is necessary to prove the following lemma.

Lemma. For $y(x, \eta)$ and $z(x, \eta)$ satisfying equations (3) the following results hold.

(i) $y \neq$ const. $\dot{y}^2 - y'^2 = 0 \Rightarrow a(\eta) =$ const.

(ii) $\dot{y}^2 - y'^2 \neq 0 \Leftrightarrow \dot{y}z' \neq y'\dot{z}$.

(iii) For $\dot{y}z' \neq y'\dot{z}$, equations (3) can be rewritten as follows:

$$1 + (a^2 x_z + x_y)^2 = (a^2\eta_z + \eta_y)^2 \tag{4a}$$

$$1 + (a^2 x_z - x_y)^2 = (a^2\eta_z - \eta_y)^2 \tag{4b}$$

where y and z have been treated as independent variables, while x and η have been treated as function of y and z, i.e., for equations (4)

$$\begin{aligned} x &= x(y, z) \\ \eta &= \eta(y, z) \\ x_z &= \partial x/\partial z \\ \eta_y &= \partial \eta/\partial y, \quad \text{etc.} \\ x_y\eta_z &\neq x_z\eta_y \end{aligned} \tag{5}$$

The above inequality is necessary because equations (3) presuppose that x and η are not functionally related.

(iv)

$$\dot{y}z' \neq y'\dot{z},\ a = 1 \Rightarrow \begin{cases} x = \phi(\alpha) + \psi(\beta) & \text{(6a)} \\ 2\eta_\alpha = \pm\sqrt{1 + 4\phi_\alpha^2} & \text{(6b)} \\ 2\eta_\beta = \pm\sqrt{1 + 4\psi_\beta^2} & \text{(6c)} \end{cases}$$

where

$$\alpha = y + z, \qquad \beta = y - z \tag{7}$$

and ϕ and ψ are arbitrary functions.

(v)

$$y' = \text{const} \Rightarrow \begin{cases} z' = \text{const} & \text{(8a)} \\ \dot{y} \text{ is a function of } \eta & \text{(8b)} \\ \dot{z} \text{ is a function of } \eta & \text{(8c)} \end{cases}$$

Proof of Part (i). In this case equations (3) reduce to

$$a^2\dot{y} = z', \qquad a^2 y' = \dot{z}$$

From the above two equations one can eliminate z by using $\partial z'/\partial\eta = \partial\dot{z}/\partial x$ to get

$$a^2\ddot{y} + 2a\dot{a}\dot{y} = a^2 y'' \tag{9}$$

As $\dot{y}^2 - y'^2 = 0 \Rightarrow \ddot{y} = y''$, one can easily conclude from (9) that $\dot{a} = 0$ (therefore $y \neq$ const) which implies $a =$ const.

Proof of Part (ii). Multplying (3a) and (3b) by $\dot{y}$ and y', respectively, and then subtracting one from other, one can trivially prove Part (ii).

Proof of Part (iii). As $\dot{y}z' \neq y'\dot{z}$, then neither y nor z is constant, nor are y and z functionally related. Hence one can treat y and z as independent variables and regard x and η as functions of y and z. Now $\dot{y}$, y', $\dot{z}$, and z' are related to x_z, x_y, η_z, and η_y by the relations

$$\begin{aligned} x_y y' + x_z z' &= 1 \\ x_y \dot{y} + x_z \dot{z} &= 0 \\ \eta_y y' + \eta_z z' &= 0 \\ \eta_y \dot{y} + \eta_z \dot{z} &= 1 \end{aligned} \tag{10}$$

Using equations (10), one get equations (4) from equations (3) after a little algebra.

Proof of Part (iv). Put $a = 1$ in equations (4) to get

$$1 + (x_z + x_y)^2 = (\eta_z + \eta_y)^2 \tag{11a}$$

$$1 + (x_x - x_y)^2 = (\eta_z - \eta_y)^2 \tag{11b}$$

Equations (11) can be rewritten as

$$2\eta_\alpha = \pm \sqrt{1 + 4x_\alpha^2} \tag{12a}$$

$$2\eta_\beta = \pm \sqrt{1 + 4x_\beta^2} \tag{12b}$$

where

$$\alpha = y + z, \qquad \beta = y - z \tag{12c}$$

Here x and η have been treated as functions of α and β, and $x_\alpha = \partial x(\alpha, \beta)/\partial\alpha$ and so on.

From equations (12a) and (12b) one can eliminate η by using $\eta_{\alpha\beta} = \eta_{\beta\alpha}$ to get

$$x_{\alpha\beta}(x_\alpha + x_\beta)(x_\alpha - x_\beta) = 0 \tag{13}$$

Equation (13) for the present case implies $x_{\alpha\beta} = 0$, which will now be proved.

If possible let $x_{\alpha\beta} \neq 0$. Then from the equations (12) and (13) there are four possibilities.

(A) $x_\alpha = x_\beta$, $\eta_\alpha = \eta_\beta$
(B) $x_\alpha = x_\beta$, $\eta_\alpha = -\eta_\beta$
(C) $x_\alpha = -x_\beta$, $\eta_\alpha = \eta_\beta$
(D) $x_\alpha = -x_\beta$, $\eta_\alpha = -\eta_\beta$

Either of (A) and (D) implies that x and η are functionally related, which is not possible here. On the other hand, (B) implies $x = x(\alpha + \beta)$ and $\eta = \eta(\alpha - \beta)$, which in view of equations (12) implies $x_\alpha =$ const, which implies $x_{\alpha\beta} = 0$. Similarly (C) also gives $x_{\alpha\beta} = 0$. So in any case one must have

$$x_{\alpha\beta} = 0 \tag{14}$$

which implies (6a). Using (6a) in equations (12), we get equations (6b) and (6c), which proves Part (iv) of the lemma.

Proof of Part (v). Now, $y' =$ const $\Rightarrow \dot{y}' = 0$, which implies (8b). From equations (3), $y' =$ const $\Rightarrow \dot{z}$ and z' are independent of $x \Rightarrow \dot{z} = 0 \Rightarrow z'$ is independent of η since z' is independent of both x and η, then $z' =$ const and $\dot{z}$ is a function of η, which implies (8a) and (8c).

The solutions are now given as follows:

I. Solutions with $y \neq const$, $\dot{y}^2 - y'^2 = 0$. Here solutions exist only if $a =$ const. The solutions are

$$(1) \quad y = f(x + \eta). \quad z = g(x + \eta) \tag{15a}$$

$$(2) \quad y = f(x - \eta). \quad z = g(x - \eta) \tag{15b}$$

where f and g are arbitrary functions. These are the solutions obtained by Vilenkin.[(1)]

II. Solutions with $\dot{y}^2 - y'^2 \neq 0$. Here equations (2) or (3) have been solved only for $a =$ const, which without loss of generality gives $a = 1$. The solutions are given by

$$x = \phi(\alpha) + \psi(\beta) \tag{16a}$$

$$\eta = \pm \int \sqrt{1 + 4\phi_\alpha^2}\, d\alpha \pm \int \sqrt{1 + 4\psi_\beta^2}\, d\beta \tag{16b}$$

where ϕ and ψ are arbitrary functions of α and β, and α, β are given by (12c).

III. Solutions with $y' = const$. In this case the solution of equations (3) as well as of equation (2) is given by

$$y(x, \eta) = c_3 + c_1 x \pm c_2 \sqrt{1 + c_1^2} \int \frac{d\eta}{\sqrt{a^4 + c_2^2}} \qquad)(17)$$

where $C_{i=1,2,3}$ are constants of integration. The solution (17) is valid for arbitrary $a(\eta)$.

Thus we have solve the equations of a cosmic string generally for the flat-space-time case. Such solutions are given by equations (15) and (16). The solution given by equation (15) was already obtained by Vilenkin.[(1)] For the expanding-universe case we have solved the equation of a cosmic string with the assumption $y' =$ const. This solution is given by (17), which is valid for arbitrary $a(\eta)$.

REFERENCES

1. A. Vilenkin, *Phys. Rev. D* **24**, 2082 (1981).